Math 221 - Fall 2015 - Homework 4 Solutions

**Problem 1**. Let and .

1. Compare the Taylor polynomials for at zand at .
2. What -degree Taylor polynomial about is needed such that for ?
3. Let be given by Taylor’s Formula with Remainder. Show that as .

*Solution.* Part a. Observe that

By the chain rule,

for all . It follows that the th Taylor polynomial of centered at is

Since the th Taylor polynomial of centered at is

it follows that

for all .

Part b. Let

for each , so that

for all . Note that

for all , and that

for all , and so

for all .

We let

and compute the first several values of :

It follows that , whence

for all .

Part c. Fix and observe that

where

Since for all choices of and , we see that

We now conclude that

Since the choice of was arbitrary, the proof is now complete. ◻

**Problem 2**. Fourier series is used to approximate periodic functions. Fourier cosine series, along with its counterpart Fourier sine series, is an extremely powerful tool that many disciplines—such as engineering, physics, biology, chemistry, computer science, and finance—call upon on a regular basis. Fourier series is the main object of study in the field of *Fourier analysis*.

1. Fix an integer , and let be constants. Show that
2. is a -periodic function, i.e., for all .

* (*Hint*: for each real number and every integer .)

1. Let
2. where are suitably chosen constants. Choose as follows:
   1. Let and be positive integers. Show that
   2. Let and be positive integers. Show that

Now, if we assume that there exist constants such that formula for holds, then for each integer ,

Therefore, it would be sensible to set

for each . As for the case, we observe that

Therefore, it would be reasonable to define

Setting

we see that

We call

the *Fourier cosine series* of . For each integer , we call the *th Fourier cosine coefficient* of . Unfortunately, the Fourier cosine series of a function certainly does not equal the function at all times.

1. Let on . Compute the Fourier cosine series of . Conclude that the Fourier cosine series of does not equal anywhere on , other than at .
2. Fix an integer , and let be constants. Show that
3. is an even function, i.e., for all .
4. Let on (one period of the so-called “triangle wave”).
   1. Compute the Fourier cosine series of .
   2. Since
   3. we see that
   4. Show that

*Solution.* Part a. Observe that

for all choices of . It follows that is -periodic.

Part b.(i)

Similarly, the angle difference formula implies that

It follows that

Part b.(ii) Problem 2.2 implies that

If , then both integrals are 0. If , then

but

Part c. Since is odd, and is even, it follows that is odd. We conclude that

for all . The Fourier cosine series of is thus given by

and this cannot equal for any .

Part d. Observe that

for all choices of . It follows that is even.

Part e.(i) Since and are both even, is even as well. Therefore,

If , then

If , then we integrate by substitution and then integrate by parts to obtain the following:

Since

we see that

It follows that the Fourier cosine series of is

Part e.(ii) The previous problem implies that

Plugging in , we obtain the following identity:

Rearranging, we obtain

Now, we observe that

Since

it follows that

We conclude that

as was to be shown. ◻